

# A NOTE ON FORMULAS TRANSMUTING MIXED MULTIPLICITIES <sup>1</sup>

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**ABSTRACT:** This paper establishes mixed multiplicity formulas concerning the relationship between mixed multiplicities of modules and mixed multiplicities of rings via rank of modules.

## 1. Introduction

It has long been known that the mixed multiplicity is an important invariant of Algebraic Geometry and Commutative Algebra. In 1973, Risler-Teissier [13] showed that each mixed multiplicity of  $\mathfrak{m}$ -primary ideals is the multiplicity of an ideal generated by a superficial sequence. For the case of arbitrary ideals, Viet [17] in 2000 characterized mixed multiplicities as the Hilbert-Samuel multiplicity via (FC)-sequences. In past years, the positivity and the relationship between mixed multiplicities and the Hilbert-Samuel multiplicity have attracted much attention (see e.g. [2, 3, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 19]).

In a recent paper [20], by a new approach, the authors gave the additivity and reduction formulas for mixed multiplicities of multi-graded modules and mixed multiplicities of arbitrary ideals; and they also showed that mixed multiplicities of arbitrary ideals are additive on exact sequences.

As a continuation, this paper gives mixed multiplicity formulas concerning the relationship between mixed multiplicities of modules and mixed multiplicities of rings via rank of modules.

Throughout the paper, denote by  $(R, \mathfrak{n})$  an artinian local ring with maximal ideal

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**n.** Let  $S = \bigoplus_{n_1, \dots, n_s \geq 0} S_{(n_1, \dots, n_s)}$  ( $s > 0$ ) be a finitely generated standard  $\mathbb{N}^s$ -graded algebra over  $R$  and let  $M = \bigoplus_{n_1, \dots, n_s \geq 0} M_{(n_1, \dots, n_s)}$  be a finitely generated  $\mathbb{N}^s$ -graded  $S$ -module such that  $M_{(n_1, \dots, n_s)} = S_{(n_1, \dots, n_s)} M_{(0, \dots, 0)}$  for all  $n_1, \dots, n_s \geq 0$ . We define  $S_{++}$  to be  $\bigoplus_{n_1, \dots, n_s > 0} S_{(n_1, \dots, n_s)}$ . Denote by  $\text{Proj } S$  the set of the homogeneous prime ideals of  $S$  which do not contain  $S_{++}$ . Set

$$\text{Supp}_{++} M = \left\{ P \in \text{Proj } S \mid M_P \neq 0 \right\}$$

and  $\dim \text{Supp}_{++} M = m$ . By [4, Theorem 4.1],  $\ell_R[M_{(n_1, \dots, n_s)}]$  is a polynomial of degree  $m$  for all large  $n_1, \dots, n_s$ . The terms of total degree  $m$  in this polynomial have the form

$$\sum_{k_1 + \dots + k_s = m} e(M; k_1, \dots, k_s) \frac{n_1^{k_1} \dots n_s^{k_s}}{k_1! \dots k_s!}.$$

Then  $e(M; k_1, \dots, k_s)$  is called the *mixed multiplicity of  $M$  of type  $(k_1, \dots, k_s)$*  [4]. In the case that  $(A, \mathfrak{m})$  is a noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $J$  is an  $\mathfrak{m}$ -primary ideal,  $I_1, \dots, I_s$  are ideals of  $A$ ,  $N$  is a finitely generated  $A$ -module, it is easily seen that

$$F_J(J, I_1, \dots, I_s; N) = \bigoplus_{n_0, n_1, \dots, n_s \geq 0} \frac{J^{n_0} I_1^{n_1} \dots I_s^{n_s} N}{J^{n_0+1} I_1^{n_1} \dots I_s^{n_s} N}$$

is a finitely generated multi-graded  $F_J(J, I_1, \dots, I_s; A)$ -module. Then the mixed multiplicity of  $F_J(J, I_1, \dots, I_s; N)$  of type  $(k_0, k_1, \dots, k_s)$  is denoted by

$$e_A(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; N)$$

and called the *mixed multiplicity of  $N$  with respect to ideals  $J, I_1, \dots, I_s$  of type  $(k_0 + 1, k_1, \dots, k_s)$*  (see [9, 16]).

Set  $\mathbf{k} = k_1, \dots, k_s$ ;  $|\mathbf{k}| = k_1 + \dots + k_s$ ;  $\mathbf{I} = I_1, \dots, I_s$ ;  $\mathbf{I}^{\mathbf{n}} = I_1^{n_1}, \dots, I_s^{n_s}$ ; and  $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \dots, I_s^{[k_s]}$ .

Let  $\mathcal{A}$  be a commutative ring,  $\mathcal{M}$  an  $\mathcal{A}$ -module, and  $Q$  be the total ring of fractions of  $\mathcal{A}$ . Then  $\mathcal{M}$  has *rank*  $r$  if  $\mathcal{M} \otimes Q$  is a free  $Q$ -module of rank  $r$ .

Then we first obtain the following result for mixed multiplicities of multi-graded modules.

**Theorem 3.1.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^s$ -graded algebra over an artinian local ring  $R$  and let  $M$  be a finitely generated  $\mathbb{N}^s$ -graded  $S$ -module of positive rank such that  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\text{Ann}_S M}$ . Then*

$$e(M; \mathbf{k}) = e(S; \mathbf{k}) \text{rank}_S M.$$

The following theorem and its corollaries are generalizations of classical results on the Hilbert-Samuel multiplicity (see e.g. [1, Corollary 4.7.9] and [5, Corollary 11.2.6]).

**Theorem 3.4.** *Let  $(A, \mathfrak{m})$  be a noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $N$  be a finitely generated  $A$ -module of positive rank. Assume that  $I = I_1 \cdots I_s$  is not contained in  $\sqrt{\text{Ann}_A N}$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \text{rank}_A N.$$

Next, we establish formulas concerning the relationship between mixed multiplicities of a noetherian local ring  $A$  and mixed multiplicities of module-finite extension rings of  $A$  of positive rank that are generalizations of [5, Theorem 11.2.7] to the mixed multiplicities of ideals. These results are started by the following theorem.

**Theorem 3.9.** *Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $B$  be a module-finite extension ring of  $A$  of positive rank. Assume that  $I = I_1 \cdots I_s$  is an ideal of positive height. Denote by  $\Pi$  the set of all maximal ideals  $Q$  of  $B$  such that  $\dim B_Q = d$ . Set  $\mathbf{I}B_Q = I_1 B_Q, \dots, I_s B_Q$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) = \sum_{Q \in \Pi} \frac{e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I}B_Q^{[\mathbf{k}]}; B_Q)[B/Q : k]}{\text{rank}_A B}.$$

The above three theorems yield interesting consequences for the cases of domains and finite extension algebras (see Corol. 3.2; Corol. 3.3; Corol. 3.6; Corol. 3.7; Corol. 3.8; Corol. 3.12; Corol. 3.13, Section 3).

Our approach is based on results in [20] and ideas in proofs of classical results on the Hilbert-Samuel multiplicity.

This paper is divided into three sections. Section 2 is devoted to the discussion of mixed multiplicities of multi-graded modules and the multiplicity of multi-graded Rees modules, and obtains a multiplicity formula of multi-graded Rees modules (see Proposition 2.1, Section 2). Section 3 proves mixed multiplicity formulas concerning the relationship between mixed multiplicities of modules and mixed multiplicities of rings via rank of modules.

## 2. Mixed multiplicities of multi-graded modules

This section defines mixed multiplicities of multi-graded modules and mixed multiplicities of ideals with respect to modules over local rings; and some other objects that will be used in the paper.

Set  $\mathbf{k} = k_1, \dots, k_s$ ;  $\mathbf{k}! = k_1! \cdots k_s!$ ;  $|\mathbf{k}| = k_1 + \cdots + k_s$ ;  $\mathbf{n}^{\mathbf{k}} = n_1^{k_1} \cdots n_s^{k_s}$ . Assume that  $S_{(1, \dots, 1)} \not\subseteq \sqrt{\text{Ann}_S M}$  and  $\dim \text{Supp}_{++} M = m$ , then by [4, Theorem 4.1],  $\ell_R[M_{\mathbf{n}}]$  is a polynomial of degree  $m$  for all large  $\mathbf{n}$ . The terms of total degree  $m$  in this polynomial have the form

$$\sum_{|\mathbf{k}|=m} e(M; \mathbf{k}) \frac{\mathbf{n}^{\mathbf{k}}}{\mathbf{k}!}.$$

Then  $e(M; \mathbf{k})$  are non-negative integers not all zero, called the *mixed multiplicity of  $M$  of the type  $\mathbf{k}$*  [4].

From now on, denote by  $P_M(\mathbf{n})$  the Hilbert polynomial of the Hilbert function  $\ell_R[M_{\mathbf{n}}]$ .

By [18, Proposition 2.7],  $\text{Supp}_{++} M = \emptyset$  (i.e.,  $S_{(1,1,\dots,1)} \subseteq \sqrt{\text{Ann}_S M}$ ) if and only if  $M_{\mathbf{n}} = 0$  for all  $\mathbf{n} \gg 0$ . If we assign  $\dim \text{Supp}_{++} M = -\infty$  to the case that  $\text{Supp}_{++} M = \emptyset$  and the degree  $-\infty$  to the zero polynomial then we have

$$\deg P_M(\mathbf{n}) = \dim \text{Supp}_{++} M.$$

Let  $(A, \mathfrak{m})$  be a noetherian local ring with maximal ideal  $\mathfrak{m}$ , residue field  $k = A/\mathfrak{m}$  and let  $N$  be a finitely generated  $A$ -module. Let  $I_1, \dots, I_s$  be ideals of  $A$  such that  $I_1 \cdots I_s$  is not contained in  $\sqrt{\text{Ann}_A N}$  and  $J$  an  $\mathfrak{m}$ -primary ideal. Put  $\mathbf{I} = I_1, \dots, I_s$ ;  $\mathbf{I}^{\mathbf{n}} = I_1^{n_1}, \dots, I_s^{n_s}$ ;  $\mathbf{I}^{[\mathbf{k}]} = I_1^{[k_1]}, \dots, I_s^{[k_s]}$ . Denote by

$$\mathfrak{R}(\mathbf{I}; A) = \bigoplus_{n_1, \dots, n_s \geq 0} I_1^{n_1} \cdots I_s^{n_s}$$

the multi-Rees algebra of ideals  $I_1, \dots, I_s$  and by

$$\mathfrak{R}(\mathbf{I}; N) = \bigoplus_{n_1, \dots, n_s \geq 0} I_1^{n_1} \cdots I_s^{n_s} N$$

the multi-Rees module of ideals  $I_1, \dots, I_s$  with respect to  $N$ . Set

$$F_J(J, \mathbf{I}; A) = \bigoplus_{n_0, n_1, \dots, n_s \geq 0} \frac{J^{n_0} I_1^{n_1} \cdots I_s^{n_s}}{J^{n_0+1} I_1^{n_1} \cdots I_s^{n_s}}$$

and

$$F_J(J, \mathbf{I}; N) = \bigoplus_{n_0, n_1, \dots, n_s \geq 0} \frac{J^{n_0} I_1^{n_1} \dots I_s^{n_s} N}{J^{n_0+1} I_1^{n_1} \dots I_s^{n_s} N}.$$

Then  $F_J(J, \mathbf{I}; A)$  is a finitely generated standard multi-graded algebra over an artinian local ring  $A/J$  and  $F_J(J, \mathbf{I}; N)$  is a finitely generated multi-graded  $F_J(J, \mathbf{I}; A)$ -module. Set  $I = I_1 \dots I_s$  and  $\dim \frac{N}{0_N : I^\infty} = q$ . Then by [17, Proposition 3.1] (see [9, Proposition 3.1]), we get

$$\deg P_{F_J(J, \mathbf{I}; N)}(n_0, \mathbf{n}) = q - 1.$$

Put

$$e(F_J(J, \mathbf{I}; N); k_0, k_1, \dots, k_s) = e(J^{[k_0+1]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; N) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$$

with  $k_0 + |\mathbf{k}| = q - 1$ . Then  $e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N)$  is called the *mixed multiplicity of  $N$  with respect to ideals  $J, \mathbf{I}$  of type  $(k_0 + 1, \mathbf{k})$*  (see [9, 16]). Set

$$\mathfrak{R}(\mathbf{I}; A)_+ = \bigoplus_{n_1 + \dots + n_s > 0} I_1^{n_1} \dots I_s^{n_s}; \mathfrak{J} = (J, \mathfrak{R}(\mathbf{I}; A)_+);$$

$$\mathfrak{R}(\mathbf{I}_i; N) = \mathfrak{R}(I_1, \dots, I_{i-1}, I_{i+1}, \dots, I_s; N); \mathfrak{J}_i = (J, \mathfrak{R}(\mathbf{I}_i; A)_+)$$

and  $\overline{N} = \frac{N}{0_N : I^\infty}$ . Then by [20, Theorem 5.2(ii) and Note 5.4], we get the following.

**Proposition 2.1.** We have  $\sum_{k_0 + |\mathbf{k}| = q-1; k_i=0} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e(\mathfrak{J}_i; \mathfrak{R}(\mathbf{I}_i; \overline{N}))$ .

### 3. Some formulas for mixed multiplicities

In this section, we prove the mixed multiplicity formulas concerning the relationship between mixed multiplicities of modules and mixed multiplicities of rings via rank of modules.

First, we have the following result for  $\mathbb{N}^s$ -graded  $S$ -modules.

**Theorem 3.1.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^s$ -graded algebra over an artinian local ring  $R$  and let  $M$  be a finitely generated  $\mathbb{N}^s$ -graded  $S$ -module of positive rank such that  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\text{Ann}_S M}$ . Then*

$$e(M; \mathbf{k}) = e(S; \mathbf{k}) \text{rank}_S M.$$

**Proof.** Since  $S_{(1,1,\dots,1)} \not\subseteq \sqrt{\text{Ann}_S M}$ , it follows that  $\text{Supp}_{++} M \neq \emptyset$ . Let  $\Lambda$  be the set of all homogeneous prime ideals  $P$  of  $S$  such that  $P \in \text{Supp}_{++} M$  and  $\dim \text{Proj } (S/P) = \dim \text{Supp}_{++} M$ . Then by [20, Theorem 3.1], we have

$$e(M; \mathbf{k}) = \sum_{P \in \Lambda} \ell(M_P) e(S/P; \mathbf{k}).$$

Denote by  $T$  the total ring of fractions of  $S$ . Since  $M$  has positive rank,  $M \otimes T \cong T^r$  is a free  $T$ -module of rank  $r > 0$ . Hence

$$M_P \cong (M \otimes T)_P \cong T_P^r \cong S_P^r \neq 0$$

for any  $P \in \text{Ass } S$ . So  $\text{Ass } S \subseteq \text{Supp } M$ . Since  $S_{(1,1,\dots,1)} \not\subseteq \sqrt{\text{Ann}_S M}$ , it follows that  $\emptyset \neq \Lambda \subseteq \text{Min}(S/\text{Ann}_S M)$  by [4, Lemma 1.1]. Consequently  $\Lambda \subseteq \text{Ass } S$ . From this it follows that  $M_P \cong S_P^r$  for any  $P \in \Lambda$ . Therefore  $\ell(M_P) = r\ell(S_P)$  for any  $P \in \Lambda$ . This fact yields

$$e(M; \mathbf{k}) = \sum_{P \in \Lambda} r\ell(S_P) e(S/P; \mathbf{k}) = \text{rank}_S M \sum_{P \in \Lambda} \ell(S_P) e(S/P; \mathbf{k}).$$

Remember that  $\text{Ass } S \subseteq \text{Supp } M$ . Hence we have  $\dim \text{Proj } S = \dim \text{Supp}_{++} M$ . So in this case,  $\Lambda$  is also the set of all homogeneous prime ideals  $P$  of  $S$  such that  $P \in \text{Proj } S$  and

$$\dim \text{Proj } (S/P) = \dim \text{Proj } S.$$

Therefore,  $\sum_{P \in \Lambda} \ell(S_P) e(S/P; \mathbf{k}) = e(S; \mathbf{k})$  by [20, Theorem 3.1]. Thus

$$e(M; \mathbf{k}) = e(S; \mathbf{k}) \text{rank}_S M. \blacksquare$$

In the case that  $S$  is a finitely generated standard graded domain over a field with field of fractions  $K$  and  $M$  is a finitely generated graded  $S$ -module, we have that  $K$  is the total ring of fractions of  $S$  and  $K \otimes M$  is a finitely generated  $K$ -vector space. Hence  $M$  has rank. And if  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\text{Ann}_S M}$  then  $M$  has positive rank:  $\text{rank}_S M = \dim_K(K \otimes M)$ . We obtain the following result.

**Corollary 3.2.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^s$ -graded domain over a field with field of fractions  $K$  and let  $M$  be a finitely generated  $\mathbb{N}^s$ -graded  $S$ -module such that  $S_{(1,1,\dots,1)}$  is not contained in  $\sqrt{\text{Ann}_S M}$ . Then  $K \otimes M$  is a finitely generated  $K$ -vector space and*

$$e(M; \mathbf{k}) = e(S; \mathbf{k}) \dim_K(K \otimes M).$$

Now, assume that  $S$  is a finitely generated standard graded domain over a field with field of fractions  $K$  and  $\mathcal{S}$  is a module-finite extension domain of  $S$  with field

of fractions  $\mathcal{K}$ . Denote by  $[\mathcal{K} : K]$  the degree of  $\mathcal{K}$  over  $K$ . Then  $K \otimes \mathcal{S} \cong K^{[\mathcal{K} : K]}$ . So  $\dim_K(K \otimes \mathcal{S}) = [\mathcal{K} : K]$ . Hence by Corollary 3.2, these facts yield:

**Corollary 3.3.** *Let  $S$  be a finitely generated standard  $\mathbb{N}^s$ -graded domain over a field with field of fractions  $K$ . And suppose that  $\mathcal{S}$  is a module-finite extension standard  $\mathbb{N}^s$ -graded domain of  $S$  with field of fractions  $\mathcal{K}$ . Then  $[\mathcal{K} : K]$  is finite and*

$$e(\mathcal{S}; \mathbf{k}) = e(S; \mathbf{k})[\mathcal{K} : K].$$

In the case of mixed multiplicities of ideals we have the following result.

**Theorem 3.4.** *Let  $(A, \mathfrak{m})$  be a noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $N$  be a finitely generated  $A$ -module of positive rank. Assume that  $I = I_1 \cdots I_s$  is not contained in  $\sqrt{\text{Ann}_A N}$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \text{rank}_A N.$$

**Proof.** It is natural to suppose that one can prove Theorem 3.4 by using Theorem 3.1. But in fact, this approach seems inconvenient. The following proof of Theorem 3.4 is independent of Theorem 3.1.

Set  $\bar{N} = \frac{N}{0_N : I^\infty}$  and  $\bar{A} = \frac{A}{0 : I^\infty}$ . Denote by  $\Pi$  the set of all prime ideals  $\mathfrak{p}$  of  $A$  such that  $\mathfrak{p} \in \text{Min}(A/\text{Ann}_A \bar{N})$  and  $\dim A/\mathfrak{p} = \dim \bar{N}$ . By [20, Theorem 3.2] we get

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = \sum_{\mathfrak{p} \in \Pi} \ell(N_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A/\mathfrak{p}).$$

Denote by  $D$  the total ring of fractions of  $A$ . Since  $N$  has positive rank  $r > 0$ ,  $N \otimes D$  is a free  $D$ -module of rank  $r > 0$ . Therefore  $N \otimes D \cong D^r$ . Hence for any  $\mathfrak{p} \in \text{Ass} A$ , we have and  $N_{\mathfrak{p}} \cong (N \otimes D)_{\mathfrak{p}} \cong D_{\mathfrak{p}}^r \cong A_{\mathfrak{p}}^r \neq 0$ . So  $\text{Ass} A \subseteq \text{Supp} N$ . From this it follows that

$$\dim \left[ \frac{A}{0 : I^\infty} \right] = \dim \left[ \frac{A}{\text{Ann}_A N : I^\infty} \right].$$

Note that  $\text{Ann}_A \bar{N} = \text{Ann}_A N : I^\infty$ ,

$$\dim \bar{N} = \dim \left[ \frac{A}{\text{Ann}_A N : I^\infty} \right] = \dim \bar{A}.$$

By [20, Remark 3.3], we have

$$\Pi = \left\{ \mathfrak{p} \in \text{Min} \left( \frac{A}{\text{Ann}_A N} \right) \mid \mathfrak{p} \not\supseteq I \text{ and } \dim A/\mathfrak{p} = \dim \bar{N} \right\}.$$

On the other hand  $\text{Ass}A \subseteq \text{Supp}N$ ,

$$\Pi = \left\{ \mathfrak{p} \in \text{Min}A \mid \mathfrak{p} \not\supseteq I \text{ and } \dim A/\mathfrak{p} = \dim \overline{N} \right\}.$$

Since  $\dim \overline{N} = \dim \bar{A}$  and

$$\left\{ \mathfrak{p} \in \text{Min}A \mid \mathfrak{p} \not\supseteq I \right\} = \text{Min} \left[ \frac{A}{0 : I^\infty} \right],$$

we get

$$\begin{aligned} \Pi &= \left\{ \mathfrak{p} \in \text{Min} \left[ \frac{A}{0 : I^\infty} \right] \mid \dim A/\mathfrak{p} = \dim \overline{N} \right\} \\ &= \left\{ \mathfrak{p} \in \text{Min} \bar{A} \mid \dim A/\mathfrak{p} = \dim \bar{A} \right\}. \end{aligned}$$

It is easily seen that  $\Pi \subseteq \text{Ass}A$ ,

$$N_{\mathfrak{p}} \cong (N \otimes D)_{\mathfrak{p}} \cong D_{\mathfrak{p}}^r \cong A_{\mathfrak{p}}^r$$

for any  $\mathfrak{p} \in \Pi$ . Hence  $\ell(N_{\mathfrak{p}}) = r\ell(A_{\mathfrak{p}})$  for any  $\mathfrak{p} \in \Pi$ . So we have

$$e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = r \sum_{\mathfrak{p} \in \Pi} \ell(A_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A/\mathfrak{p}).$$

Now since

$$\Pi = \left\{ \mathfrak{p} \in \text{Min} \bar{A} \mid \dim A/\mathfrak{p} = \dim \bar{A} \right\},$$

it follows that

$$\sum_{\mathfrak{p} \in \Pi} \ell(A_{\mathfrak{p}}) e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A/\mathfrak{p}) = e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A)$$

by [20, Theorem 3.2]. Hence we obtain

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \text{rank}_A N. \blacksquare$$

**Remark 3.5.** It would be desirable to obtain Theorem 3.4 as a consequence of Theorem 3.1, that is, to prove that

$$\text{rank}_{F_J(J, \mathbf{I}; A)} F_J(J, \mathbf{I}; N) = \text{rank}_A N.$$

Note that if  $A$  is a domain with field of fractions  $K$  then  $N$  has rank and  $\text{rank}_A N = \dim_K(K \otimes N)$ . Then as an immediate consequence of Theorem 3.4, we get the following.



**Corollary 3.6.** *Let  $(A, \mathfrak{m})$  be a noetherian local domain with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , and field of fractions  $K$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $N$  be a finitely generated  $A$ -module. Assume that  $I = I_1 \cdots I_s$  is not contained in  $\sqrt{\text{Ann}_A N}$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \dim_K(K \otimes N).$$

In particular, if  $A$  is a domain with field of fractions  $K$  and  $\mathcal{A}$  is a module-finite extension domain of  $A$  with field of fractions  $\mathcal{K}$ . Then  $K \otimes \mathcal{A} \cong K^{[\mathcal{K}:K]}$ . Hence  $\dim_K(K \otimes \mathcal{A}) = [\mathcal{K} : K]$ . Therefore by Corollary 3.6, we have the following result.

**Corollary 3.7.** *Let  $(A, \mathfrak{m})$  be a noetherian local domain with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , and field of fractions  $K$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $\mathcal{A}$  be a module-finite extension domain of  $A$  with field of fractions  $\mathcal{K}$ . Assume that  $I = I_1 \cdots I_s \neq 0$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; \mathcal{A}) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A)[\mathcal{K} : K].$$

Keep the conditions as in Theorem 3.4, then we have  $\dim \left[ \frac{A}{0 : I^\infty} \right] = \dim \left[ \frac{N}{0_N : I^\infty} \right]$ . Hence by [20, Corollary 2.5] which is a generalized version of [16, Theorem 1.4] and [4, Theorem 4.4],  $e\left((J, \mathfrak{R}(\mathbf{I}; A)_+); \mathfrak{R}(\mathbf{I}; \frac{A}{0 : I^\infty})\right) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A)$  and

$$e\left((J, \mathfrak{R}(\mathbf{I}; A)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N).$$

Therefore

$$\begin{aligned} e\left((J, \mathfrak{R}(\mathbf{I}; A)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) &= \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; N) \\ &= \left[ \sum_{k_0 + |\mathbf{k}| = q-1} e(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \right] \text{rank}_A N \\ &= e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{A}{0 : I^\infty})\right) \text{rank}_A N \end{aligned}$$

by Theorem 3.4. Consequently,

$$e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{A}{0 : I^\infty})\right) \text{rank}_A N.$$

These facts yield:

**Corollary 3.8.** *Let  $(A, \mathfrak{m})$  be a noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $N$  be a finitely generated  $A$ -module of positive rank. Assume that  $I = I_1 \cdots I_s$  is not contained in  $\sqrt{\text{Ann}_A N}$ . Then we have*

$$e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{N}{0_N : I^\infty})\right) = e\left((J, \mathfrak{R}(\mathbf{I}; R)_+); \mathfrak{R}(\mathbf{I}; \frac{A}{0 : I^\infty})\right) \text{rank}_A N.$$

Next, we give formulas on the relationship between mixed multiplicities of a noetherian local ring  $A$  and mixed multiplicities of a module-finite extension ring of  $A$  of positive rank that are generalizations of [5, Theorem 11.2.7] to the mixed multiplicities of ideals.

**Theorem 3.9.** *Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $B$  be a module-finite extension ring of  $A$  of positive rank. Assume that  $I = I_1 \cdots I_s$  is an ideal of positive height. Denote by  $\coprod$  the set of all maximal ideals  $Q$  of  $B$  such that  $\dim B_Q = d$ . Set  $\mathbf{I}B_Q = I_1 B_Q, \dots, I_s B_Q$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]; A}) = \sum_{Q \in \coprod} \frac{e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I}B_Q^{[\mathbf{k}]; B_Q})[B/Q : k]}{\text{rank}_A B}.$$

**Note 3.10:** Let  $F$  be a  $B$ -module. Then  $F$  is also an  $A$ -module. Assume that  $F$  is an  $A$ -module of finite length. Since  $B$  is a module-finite extension ring of  $A$ ,  $F$  is also a  $B$ -module of finite length. Assume that  $\ell_B(F) = n$ . Set  $\coprod = \text{Ass}_B F$ . Then there exists a composition series of  $B$ -module  $F$  of length  $n$  :

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = F$$

where  $F_i/F_{i-1} \cong B/P_i$ ,  $P_i$  maximal ( $1 \leq i \leq n$ ). And in this case,  $\coprod = \{P_1, \dots, P_n\}$ . We have

$$\ell_A(F) = \sum_{i=1}^n \ell_A(F_i/F_{i-1}) = \sum_{i=1}^n \ell_A(B/P_i).$$

Hence  $\ell_A(F)$  is a sum of all the  $\ell_A(B/P)$  for  $P \in \coprod$ , counted as many times as  $B/P$  appears as some  $B/P_i$ . This number is exactly the length of  $B_P$ -module  $F_P$ . So

$$\ell_A(F) = \sum_{P \in \coprod} \ell_A(B/P) \ell_{B_P}(F_P).$$

Because

$$\begin{aligned}\ell_A(B/P) &= \dim_k(B/P) = [B/P : k], \\ \ell_A(F) &= \sum_{P \in \mathbb{I}} \ell_{B_P}(F_P)[B/P : k].\end{aligned}\tag{1}$$

Set  $\mathbf{I}B = I_1B, \dots, I_sB$ . Then we have  $\mathfrak{R}(\mathbf{I}B; B) = \mathfrak{R}(\mathbf{I}; B)$ . Since  $B$  is a module-finite extension ring of  $A$ ,  $\mathfrak{R}(\mathbf{I}B; B)$  is a module-finite extension ring of  $\mathfrak{R}(\mathbf{I}; A)$ . Put  $\mathfrak{N} = (\mathfrak{m}, \mathfrak{R}(\mathbf{I}; A)_+)$ . The notion

$$e_{\mathfrak{R}(\mathbf{I}; A)}((J, \mathfrak{R}(\mathbf{I}; A)_+); \mathfrak{R}(\mathbf{I}B; B))$$

will mean the Hilbert-Samuel multiplicity:

$$e_{\mathfrak{R}(\mathbf{I}; A)_{\mathfrak{N}}}((J, \mathfrak{R}(\mathbf{I}; A)_+)_{\mathfrak{N}}; \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{N}}).$$

The proof of Theorem 3.9 is based on the following lemma.

**Lemma 3.11.** *Keeping the notation of Theorem 3.9, denote by  $\Omega$  the set of all maximal homogeneous ideals  $\mathfrak{M}$  of  $\mathfrak{R}(\mathbf{I}B; B)$  such that  $\dim \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}} = \dim \mathfrak{R}(\mathbf{I}; A)$ . Put  $\mathfrak{J} = (J, \mathfrak{R}(\mathbf{I}; A)_+)$ ;  $\mathfrak{J}_{\mathfrak{M}} = \mathfrak{J}\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}$ . Then*

$$e_{\mathfrak{R}(\mathbf{I}; A)}(\mathfrak{J}; \mathfrak{R}(\mathbf{I}B; B)) = \sum_{\mathfrak{M} \in \Omega} e_{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}(\mathfrak{J}_{\mathfrak{M}}; \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}) \left[ \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} : k \right].$$

**Proof.** Recall that  $\mathfrak{N} = (\mathfrak{m}, \mathfrak{R}(\mathbf{I}; A)_+)$ . Denote by  $\Gamma$  the set of all maximal homogeneous ideals of  $\mathfrak{R}(\mathbf{I}B; B)$ . Note that  $\mathfrak{N} \subseteq \mathfrak{M}$  for any  $\mathfrak{M} \in \Gamma$ . Since  $B$  is a module-finite extension ring of  $A$ ,  $\mathfrak{R}(\mathbf{I}B; B)$  is a module-finite extension ring of  $\mathfrak{R}(\mathbf{I}; A)$ . It is clear that

$$\frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)}$$

is an artinian  $\mathfrak{R}(\mathbf{I}; A)$ -module, this is also an artinian  $\mathfrak{R}(\mathbf{I}; B)$ -module. Therefore

$$\Gamma = \text{Ass}_{\mathfrak{R}(\mathbf{I}B; B)} \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)}.$$

Consequently, by (1) of Note 3.10,

$$\ell_{\mathfrak{R}(\mathbf{I}; A)_{\mathfrak{N}}} \frac{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{N}}}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{N}}} = \sum_{\mathfrak{M} \in \Gamma} \ell_{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \frac{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \left[ \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} : k \right]. \tag{2}$$

Recall that  $\mathfrak{R}(\mathbf{I}B; B)$  is a module-finite extension ring of  $\mathfrak{R}(\mathbf{I}; A)$ ,

$$\dim \mathfrak{R}(\mathbf{I}B; B) = \dim \mathfrak{R}(\mathbf{I}; A).$$

On the other hand  $\text{ht} I > 0$ ,  $\dim \mathfrak{R}(\mathbf{I}; A) = \dim A + s = d + s$ . It is easily seen that  $\text{Ann}_A B = 0$ ,  $\dim_A B = \dim A = d$ . Since  $\text{Ann}_A B = 0$  and  $\text{ht} I > 0$ ,

$$\dim_{\mathfrak{R}(\mathbf{I}; A)} \mathfrak{R}(\mathbf{I}B; B) = \dim_A B + s = d + s.$$

So

$$\dim \mathfrak{R}(\mathbf{I}; A) = \dim_{\mathfrak{R}(\mathbf{I}; A)} \mathfrak{R}(\mathbf{I}B; B) = \dim \mathfrak{R}(\mathbf{I}B; B) = d + s. \quad (3)$$

Therefore

$$e_{\mathfrak{R}(\mathbf{I}; A)}(\mathfrak{J}; \mathfrak{R}(\mathbf{I}B; B)) = \lim_{n \rightarrow \infty} \frac{(d+s)!}{n^{d+s}} \ell_{\mathfrak{R}(\mathbf{I}; A)_{\mathfrak{M}}} \frac{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}. \quad (4)$$

Because that if  $\mathfrak{M} \in \Gamma \setminus \Omega$  then  $\dim \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}} \neq \dim \mathfrak{R}(\mathbf{I}; A)$ , by (3) it follows that

$$\dim \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}} < d + s. \quad (5)$$

By (2); (4) and (5) we have

$$\begin{aligned} e_{\mathfrak{R}(\mathbf{I}; A)}(\mathfrak{J}; \mathfrak{R}(\mathbf{I}B; B)) &= \lim_{n \rightarrow \infty} \frac{(d+s)!}{n^{d+s}} \sum_{\mathfrak{M} \in \Gamma} \ell_{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \frac{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \left[ \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} : k \right] \\ &= \lim_{n \rightarrow \infty} \frac{(d+s)!}{n^{d+s}} \sum_{\mathfrak{M} \in \Omega} \ell_{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \frac{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}{\mathfrak{J}^n \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}} \left[ \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} : k \right]. \end{aligned}$$

Thus,

$$e_{\mathfrak{R}(\mathbf{I}; A)}(\mathfrak{J}; \mathfrak{R}(\mathbf{I}B; B)) = \sum_{\mathfrak{M} \in \Omega} e_{\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}}(\mathfrak{J}_{\mathfrak{M}}; \mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}}) \left[ \frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} : k \right]. \blacksquare$$

**The proof of Theorem 3.9:** It is easily seen that there is an one-to-one correspondence between the set of maximal ideals  $\Omega$  and the set of maximal ideals  $\prod$  given by

$$\mathfrak{M} \mapsto Q = \mathfrak{M} \cap B.$$

Moreover, if  $Q = \mathfrak{M} \cap B$  then  $\frac{\mathfrak{R}(\mathbf{I}B; B)}{\mathfrak{M}} \cong \frac{B}{Q}$  is a finite extension field of  $k$  and

$$\mathfrak{R}(\mathbf{I}B; B)_{\mathfrak{M}} = \mathfrak{R}(\mathbf{I}B_Q; B_Q)_{\mathfrak{M}} \text{ and } \mathfrak{J}_{\mathfrak{M}} = (JB_Q, \mathfrak{R}(\mathbf{I}B_Q; B_Q)_+) \mathfrak{R}(\mathbf{I}B_Q; B_Q)_{\mathfrak{M}}. \quad (6)$$

Remember that  $\mathbf{I}^{\mathbf{u}} = I_1^{u_1}, \dots, I_s^{u_s}$  and  $\mathbf{I}^{\mathbf{u}} B_Q = I_1^{u_1} B_Q, \dots, I_s^{u_s} B_Q$  for any  $\mathbf{u} = u_1, \dots, u_s$ . By [20, Proposition 2.4] which is a generalized version of [16, Theorem 1.4] and [4, Theorem 4.4], we have

$$e_{\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; A)}((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; A)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; B)) = \sum_{k_0 + |\mathbf{k}| = d-1} e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; B) \mathbf{u}^{\mathbf{k}}$$

and

$$\begin{aligned} & e_{\mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)}((JB_Q, \mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)) \\ &= \sum_{k_0 + |\mathbf{k}| = d-1} e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I} B_Q^{[\mathbf{k}]}; B_Q) \mathbf{u}^{\mathbf{k}}. \end{aligned}$$

Hence by Lemma 3.11 and (6), we obtain

$$\begin{aligned} & \sum_{k_0 + |\mathbf{k}| = d-1} e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; B) \mathbf{u}^{\mathbf{k}} \\ &= e_{\mathfrak{R}(\mathbf{I}^{\mathbf{u}}; A)}((J, \mathfrak{R}(\mathbf{I}^{\mathbf{u}}; A)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}} B; B)) \\ &= \sum_{Q \in \Pi} e_{\mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)}((JB_Q, \mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)_+); \mathfrak{R}(\mathbf{I}^{\mathbf{u}} B_Q; B_Q)) [B/Q : k] \\ &= \sum_{k_0 + |\mathbf{k}| = d-1} \left[ \sum_{Q \in \Pi} e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I} B_Q^{[\mathbf{k}]}; B_Q) [B/Q : k] \right] \mathbf{u}^{\mathbf{k}}. \end{aligned}$$

From this it follows that

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; B) = \sum_{Q \in \Pi} e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I} B_Q^{[\mathbf{k}]}; B_Q) [B/Q : k].$$

Remember that by Theorem 3.4,

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; B) = e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) \text{rank}_A B.$$

Thus,

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[\mathbf{k}]}; A) = \sum_{Q \in \Pi} \frac{e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I} B_Q^{[\mathbf{k}]}; B_Q) [B/Q : k]}{\text{rank}_A B}. \blacksquare$$

When  $A$  is a domain with field of fractions  $K$  and  $B$  is a module-finite extension ring of  $A$ ,  $B$  has rank and  $\text{rank}_A B = \dim_K(K \otimes B)$ . Then we obtain the following result by Theorem 3.9.

**Corollary 3.12.** *Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional noetherian local domain with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , and field of fractions  $K$ . Let  $J, I_1, \dots, I_s$  be*

ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $B$  be a module-finite extension ring of  $A$ . Denote by  $\prod$  the set of all maximal ideals  $Q$  of  $B$  such that  $\dim B_Q = d$ . Assume that  $I = I_1 \cdots I_s \neq 0$ . Then we have

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[k]}; A) = \sum_{Q \in \prod} \frac{e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I}B_Q^{[k]}; B_Q)[B/Q : k]}{\dim_K(K \otimes B)}.$$

Let  $A$  be a domain with field of fractions  $K$ , and  $B$  a module-finite extension domain of  $A$  with field of fractions  $\mathcal{K}$ . Then  $\dim_K(K \otimes B) = [\mathcal{K} : K]$ . Hence as an immediate consequence of Corollary 3.12, we give the following result.

**Corollary 3.13.** *Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional noetherian local domain with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , and field of fractions  $K$ . Let  $J, I_1, \dots, I_s$  be ideals of  $A$  with  $J$  being  $\mathfrak{m}$ -primary. Let  $B$  be a module-finite extension domain of  $A$  with field of fractions  $\mathcal{K}$ . Assume that  $I = I_1 \cdots I_s \neq 0$ . Denote by  $\prod$  the set of all maximal ideals  $Q$  of  $B$  such that  $\dim B_Q = d$ . Then we have*

$$e_A(J^{[k_0+1]}, \mathbf{I}^{[k]}; A) = \sum_{Q \in \prod} \frac{e_{B_Q}(JB_Q^{[k_0+1]}, \mathbf{I}B_Q^{[k]}; B_Q)[B/Q : k]}{[\mathcal{K} : K]}.$$

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